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Review of Wings-theory & few properties of wing-length with respect to N-equation

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Abstract: This paper contains the different properties of wing-length with respect to N-equation and using those properties it has been possible to establish a series of polynomial functions in a systematic manner that can produce a square integer only for once or twice or not at all known as square-free polynomials. For the ease of apprehension and to avoid drawing frequent references from earlier publications of 'IJSER', the whole N-equation properties of wings-theory has once again been reviewed and relevant parts are being projected as a ready reference along with all new findings. An index marking the heading points for new findings and minor corrections/modifications with respect to earlier publication is also furnished at the end of article under 'Reference'.

Key words: N-equation & Nize-equation, N_s & N_d operation, Natural constant (k), odd-zygote & mixed zygote expression, wing-length, abc & √abc-equation, prime & composite wings etc.

Introduction:

In a mixed combination of odd-even integers the expression $\alpha^2 \pm \beta^2$ can be said as positive or negative wing where α, β are the two elements of a wing. If there is no common factor in between the elements i.e. $(\alpha, \beta) = 1$ then the wing is called a prime wing and if $(\alpha, \beta) > 1$ it is a composite wing. A prime wing therefore, may indicate a prime number or a composite number both whereas a composite wing is always for a composite number. Difference of the elements is called wing length or elementary gap.
Basically there exists two kinds of prime numbers, 1st kind and 2nd kind according as $4x - 1$ form & $4x + 1$ form respectively. 1st kind prime can't be expressed as $\alpha^2 + \beta^2$ whereas 2nd kind prime is always expressible as $\alpha^2 + \beta^2$ where obviously $(\alpha, \beta) = 1$. Any wing of the form $(2x)^2 \pm 1$ can be said as positive or negative mono-wing. All other wings are general wings. Any odd composite integer is of 1st kind or 2nd kind according as it is $4x - 1$ form or $4x + 1$ form. If it is of 1st kind it must contain at least one 1st kind prime and sum of their all exponents must be odd. If it is of 2nd kind sum of exponents of all 1st kind prime must be even or zero i.e. no 1st kind prime is present & then it can be said as purely 2nd kind composite number. It is simply because of the fact that $(4x + 1)^n$ is always in the form of $4\lambda + 1$ whatever may be the nature of exponent n. but $(4x - 1)^n$ is in the form of $4\lambda + 1$ when n is even and in the form of $4\lambda - 1$ when n is odd. $(4x + 1)(4y - 1)$ is also in the form of $4\lambda - 1$.

Some usual Notations:

Here we shall deal with the positive integers only and hence, all the variables used here are of positive integers unless it is specially mentioned. Meaning of some usual notations & some new symbol are given below.

1. (a, b): greatest common factor in between a & b
2. $a|b$: a divides b
3. $a \in b$: a belongs to b & $a \notin b$: a doesn't belong to.
4. Product symbol $\prod(w_i) = w_1w_2w_3.......w_i$
5. Equality symbol $E(w_i) = w_1 = w_2 = w_3 = ... = w_i$
6. $\uparrow(N)_x$ denotes the exponent of the prime factor x present in the number N.
7. 'I' always denotes a positive integer where to distinguish even & odd I_e, I_o are used.

8. a ~ b means either a – b or b – a so that result > 0.
9. $\{f_i(x), ...\}$, $i \in I$, represents a set of functions & equals to $\{a_i, ...\}$, a set of integers of functional values common to at least for two functions where {o} means empty set.
10. w(x) generally denotes the functional form of wing length (b – a) of a wing $a^2 + b^2$ (a < b)

1. Systematic arrangement of Pythagorean triplets

1.1 Definition of a Natural Equation or simply N-equation:

$a^2 + b^2 = c^2$ where the elements a, b, c all are of positive integers is said to be a Natural Equation or simply N-equation provided its comparable equation i.e. $(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2$ has the property that α, β must be of positive integers. (a, b, c) is said to be a Pythagorean composite triplet/set or prime triplet/set according as a common factor lies among a, b, c or not.
Now, from the property $(a\mu)^2 + (b\mu)^2 = (c\mu)^2$ we can say that any prime triplet can produce infinitely many composite triplets ($a\mu$, $b\mu$, $c\mu$). But our main concern is to deal with a Pythagorean prime triplet.
Now, N-equation can be of three types:
i) $(e_1)^2 + (e_2)^2 = (e_3)^2$ ii) $(o_1)^2 + (o_2)^2 = (e)^2$ iii) $(e)^2 + (o)^2 = (o_1)^2$
where e & o denote even & odd integer respectively.
Case i) cannot be accepted as it is a composite triplet.
Case ii) cannot be accepted as $(o_1)^2 + (o_2)^2 = (2x - 1)^2 + (2y - 1)^2 = 2(2x^2 + 2y^2 - 2x - 2y + 1) = 2$(odd integer) nature and fails to form a square integer.
Case iii) can be well accepted and this can be of following two kinds.

1.2 N-equation of 1st kind and 2nd kind

To maintain the ascending order i.e. a < b < c, 1st kind is defined as odd < even < odd
$\Rightarrow 2\alpha\beta > \alpha^2 - \beta^2 \Rightarrow (\alpha/\beta) < \sqrt{2} + 1$ and 2nd kind is just its reverse.
From the comparable equation it is quite understood that α, β are the combination of even-odd integers.
For 1st kind $c - b = (\alpha - \beta)^2 = (I_o)^2 = k$ say, where k can be said as Natural constant. $c - a = 2\beta^2$ i.e. $2(I)^2$ form.
Similarly, for 2nd kind $k = c - b = 2(I)^2$ & $c - a = (I_o)^2$ & here, even < odd < odd.

1.3 N-equation of 1st kind in functional form

Here, $\{b + (2y - 1)^2 - 2x^2\}^2 + b^2 = \{b + (2y - 1)^2\}^2$ where $2x^2$ is just greater than $(2y - 1)^2$ by an integer value.
$\Rightarrow b^2 - b.4x^2 + 4x^4 - 4x^2(2y - 1)^2 = 0$ or, $b = 2x^2 \pm 2x(2y - 1)$ or, $b = 2x^2 + 4xy - 2x$, neglecting (-) sign
$\Rightarrow a = 4y^2 + 4xy - 4y - 2x + 1$ & $c = 4y^2 + 2x^2 + 4xy - 4y - 2x + 1$
$\Rightarrow (4y^2 + 4xy - 4y - 2x + 1)^2 + (2x^2 + 4xy - 2x)^2 = (4y^2 + 2x^2 + 4xy - 4y - 2x + 1)^2$
i.e. $\{f(x, y)\}^2 + \{\varphi(x, y)\}^2 = \{\psi(x, y)\}^2$ for $k = (2y - 1)^2$
For k = 1, y = 1 & as $2x^2 > (2y - 1)^2$, $x \geq 1 \Rightarrow (2x + 1)^2 + (2x^2 + 2x)^2 = (2x^2 + 2x + 1)^2$ where $x \in I$.
For k = 9, y = 2 $\Rightarrow x \geq 3 \Rightarrow (6x + 9)^2 + (2x^2 + 6x)^2 = (2x^2 + 6x + 9)^2$ where x = 3, 4, 5, …….
For a particular value of k we can change the functional expression so as to start the variable with one.
As 'a' is always linear & b, c are always quadratic, say for k = 9, $a = Az + B$ & $b = Cz^2 + Dz + E$. Obtain first three triplets by putting x = 3, 4, 5 & they are (27, 36, 45), (33, 56, 65) & (39, 80, 89). For z = 1, A + B = 27 & C + D + E = 36; for z = 2, 2A + B = 33 & 4C + 2D + E = 56 and for z = 3,
9C + 3D + E = 80. Solving them & considering variable x we get
$(6x + 21)^2 + (2x^2 + 14x + 20)^2 = (2x^2 + 14x + 29)^2$ where $x \in I$.
As the coefficient of x^2 is always 2, we can obtain the same by considering first two triplets.
i.e. $a = Az + B$, $b = 2z^2 + Cz + D$ where A + B = 27, C + D = 34 & 2A + B = 33, 2C + D = 48
Similarly, for k = 25, 49, ……. we can obtain the functional form as given below.
K = 1, $(2x + 1)^2 + (2x^2 + 2x)^2 = (2x^2 + 2x + 1)^2$ where $x \in I$ with leading triplet 3, 4, 5

K = 9, $(6x + 21)^2 + (2x^2 + 14x + 20)^2 = (2x^2 + 14x + 29)^2$ where $x \in I$ with leading triplet 27, 36, 45
K = 25, $(10x + 55)^2 + (2x^2 + 22x + 48)^2 = (2x^2 + 22x + 73)^2$ where $x \in I$ with leading triplet 65, 72, 97
K = 49, $(14x + 105)^2 + (2x^2 + 30x + 88)^2 = (2x^2 + 30x + 137)^2$, $x \in I$ with leading triplet 119, 120, 169
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1.4 N-equation of 2nd kind in functional form

Here, $k = 2y^2$ & $(b + 2y^2 – x^2)^2 + b^2 = (b + 2y^2)^2$ where $x \in I_o$ so that x^2 is just greater than $2y^2$ by an integer.
$\Rightarrow b^2 – 2bx^2 + (x^2 – 4y^2)x^2 = 0$ or, $b = x^2 \pm x\sqrt{(x^2 – x^2 + 4y^2)} = x^2 + 2xy$ neglecting (-) sign.
Accordingly, $a = x^2 + 2xy + 2y^2 – x^2 = 2y^2 + 2xy$ & $c = x^2 + 2xy + 2y^2$
$\Rightarrow (2y^2 + 2xy)^2 + (x^2 + 2xy)^2 = (x^2 + 2xy + 2y^2)^2$ & replacing x by $2x – 1$,
$\{2y^2 + 2y(2x – 1)\}^2 + \{(2x – 1)^2 + 2y(2x – 1)\}^2 = \{(2x – 1)^2 + 2y(2x – 1) + 2y^2\}^2$
For $k = 2$, $y = 1 \Rightarrow (2x – 1)^2 > 2.1^2$ i.e. $x \geq 2$ we have $(4x)^2 + (4x^2 – 1)^2 = (4x^2 + 1)^2$ where x = 2, 3, 4, ….
Here also we can modify the functional form to start with x from one by initial two triplets considering the fact that even element is linear and odd elements are quadratic with coefficient of x^2 as 4. First four functional forms are given below.
K = 2, $(4x + 4)^2 + (4x^2 + 8x + 3)^2 = (4x^2 + 8x + 5)^2$ where $x \in I$ with leading triplet 8, 15, 17
K = 8, $(8x + 12)^2 + (4x^2 + 12x + 5)^2 = (4x^2 + 12x + 13)^2$ where $x \in I$ with leading triplet 20, 21, 29
K = 18, $(12x + 36)^2 + (4x^2 + 24x + 27)^2 = (4x^2 + 24x + 45)^2$ where $x \in I$ with leading triplet 48, 55, 73
K = 32, $(16x + 72)^2 + (4x^2 + 36x + 65)^2 = (4x^2 + 36x + 97)^2$ where $x \in I$ with leading triplet 88,105,137
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It is observed that some of the triplets are composite set and some are of prime set. It is simply because of the fact that if an odd integer μ is multiplied with the elements (a, b, c) of a triplet under $k = I_o^2$ or $k = 2.I^2$ then the composite set (μa, μb, μc) must satisfy $k = (\mu I_o)^2$ or $k = 2(\mu I)^2$
Only the triplets received by $k = 2^{n-1}$ where $n \in I$ are prime sets.
To establish the functional form of N-equation for a particular value of k we therefore, need the first two triplets which can be obtained by following way.
Say, for a 1st kind N-equation $k = 25 = 5^2$.
Now if the leading set is (a, b, c) then $c = b + 25$ & $a = c – 2.4^2$ as 2.4^2 is just greater than 5^2 i.e. $a = b – 7 \Rightarrow (b – 7)^2 + b^2 = (b + 25)^2$ or, $b = 72 \Rightarrow$ 1st leading set is (65, 72, 97)
Similarly for the 2nd set $a = c – 2.5^2$ [obviously, it will be a composite set]
$\Rightarrow (b – 25)^2 + b^2 = (b + 25)^2$ or, $b = 100 \Rightarrow$ 2nd set is (75, 100, 125)

Say, for a 2nd kind N-equation $k = 18 = 2.3^2$. Here, $c = b + 18$ & $a = c – 5^2 = b – 7$
$\Rightarrow (b – 7)^2 + b^2 = (b + 18)^2$ i.e. b = b = 55 $\Rightarrow$ a = 48, c = 73 and 1st leading set is (48, 55, 73)
Similarly, for the 2nd set $a = c – 7^2 \Rightarrow (b – 31)^2 + b^2 = (b + 18)^2$ & b = 91 $\Rightarrow$ a = 60, c = 109 $\Rightarrow$ 2nd set is (60, 91, 109)

1.5 Natural equation in mixed zygote form

In mixed zygote form, N-equation can be written as,
1st kind:
$\{(y + 2x – 1)^2 – y^2\}^2 + \{2y(y + 2x – 1)\}^2 = \{(y + 2x – 1)^2 + y^2\}^2$ [as leading set]
Or, $[\{f(x, y)\}^2 – \{\varphi(y)\}^2]^2 + [2f(x, y)\varphi(y)]^2 = [\{f(x, y)\}^2 + \{\varphi(y)\}^2]^2$
Where for a particular value of k, $\{f(x, y)\} – \{\varphi(y)\}$ is constant
2nd kind:
$\{2x(x + y)\}^2 + \{(x + y)^2 – x^2\}^2 = \{(x + y)^2 + x^2\}^2$ [as leading set]
Or, $[2f(x, y)\varphi(x)]^2 + [\{f(x, y)\}^2 – \{\varphi(x)\}^2]^2 = [\{f(x, y)\}^2 + \{\varphi(x)\}^2]^2$
Where for a particular value of k $\{\varphi(x)\}$ is constant.

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1.5.1 Example Chart of 1st kind N-equation in mixed zygote form.

For k = 1 $(2^2 - 1^2)^2 + (2.2.1)^2 = (2^2 + 1^2)^2$
$(3^2 - 2^2)^2 + (2.3.2)^2 = (3^2 + 2^2)^2$
$(4^2 - 3^2)^2 + (2.4.3)^2 = (4^2 + 3^2)^2$
…………………………………
For k = 9, $(6^2 - 3^2)^2 + (2.6.3)^2 = (6^2 + 3^2)^2$
$(7^2 - 4^2)^2 + (2.7.4)^2 = (7^2 + 4^2)^2$
$(8^2 - 5^2)^2 + (2.8.5)^2 = (8^2 + 5^2)^2$
…………………………………
For k = 25, $(9^2 - 4^2)^2 + (2.9.4)^2 = (9^2 + 4^2)^2$
$(10^2 - 5^2)^2 + (2.10.5)^2 = (10^2 + 5^2)^2$
$(11^2 - 6^2)^2 + (2.11.6)^2 = (11^2 + 6^2)^2$
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1.5.2 Example Chart of 2nd kind N-equation in mixed zygote form.

For k = 2, $(2.4.1)^2 + (4^2 - 1^2)^2 = (4^2 + 1^2)^2$
$(2.6.1)^2 + (6^2 - 1^2)^2 = (6^2 + 1^2)^2$
$(2.8.1)^2 + (8^2 - 1^2)^2 = (8^2 + 1^2)^2$
…………………………………
For k = 8, $(2.5.2)^2 + (5^2 - 2^2)^2 = (5^2 + 2^2)^2$
$(2.7.2)^2 + (7^2 - 2^2)^2 = (7^2 + 2^2)^2$
$(2.9.2)^2 + (9^2 - 2^2)^2 = (9^2 + 2^2)^2$
…………………………………..
For k = 18, $(2.8.3)^2 + (8^2 - 3^2)^2 = (8^2 + 3^2)^2$
$(2.10.3)^2 + (10^2 - 3^2)^2 = (10^2 + 3^2)^2$
$(2.12.3)^2 + (12^2 - 3^2)^2 = (12^2 + 3^2)^2$
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Note: $(2d_1)^2 + (d_2)^2 \neq I_o^2$ & $v^2 + 1 \neq I_o^2$ where $d \in I_o$ & $v \in I_e$

1.6 Natural equation in odd zygote form

Let us now introduce another form of N-equation known as 'Odd zygote form'
We have $c + b = d_1^2$ & $c - b = d_2^2$ where $b \in I_e \Rightarrow c = (d_1^2 + d_2^2)/2$ & $b = (d_1^2 - d_2^2)/2$
$\Rightarrow a = \sqrt{(c^2 - b^2)} = \sqrt{\{(c + b)(c - b)} = d_1d_2 \Rightarrow \{(d_1^2 - d_2^2)/2\}^2 + (d_1d_2)^2 = (d_1^2 + d_2^2)/2$

1.6.1 Example chart of 1st kind

For k = 1, $(1.3)^2 + \{(3^2 - 1^2)/2\}^2 = \{(3^2 + 1^2)/2\}^2$
$(1.5)^2 + \{(5^2 - 1^2)/2\}^2 = \{(5^2 + 1^2)/2\}^2$
$(1.7)^2 + \{(7^2 - 1^2)/2\}^2 = \{(7^2 + 1^2)/2\}^2$
………………………………………
For k = 9, $(3.9)^2 + \{(9^2 - 3^2)/2\}^2 = \{(9^2 + 3^2)/2\}^2$
$(3.11)^2 + \{(11^2 - 3^2)/2\}^2 = \{(11^2 + 3^2)/2\}^2$
$(3.13)^2 + \{(13^2 - 3^2)/2\}^2 = \{(13^2 + 3^2)/2\}^2$
………………………………………
For k = 25, $(5.13)^2 + \{(13^2 - 5^2)/2\}^2 = \{(13^2 + 5^2)/2\}^2$
$(5.15)^2 + \{(15^2 - 5^2)/2\}^2 = \{(15^2 + 5^2)/2\}^2$
$(5.17)^2 + \{(17^2 - 5^2)/2\}^2 = \{(17^2 + 5^2)/2\}^2$
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1.6.2 Example chart of 2nd kind

For k = 2, $\{(5^2 - 3^2)/2\}^2 + (5.3)^2 = \{(5^2 + 3^2)/2\}^2$
$\{(7^2 - 5^2)/2\}^2 + (7.5)^2 = \{(7^2 + 5^2)/2\}^2$
$\{(9^2 - 7^2)/2\}^2 + (9.7)^2 = \{(9^2 + 7^2)/2\}^2$
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For k = 8, $\{(7^2 - 3^2)/2\}^2 + (7.3)^2 = \{(7^2 + 3^2)/2\}^2$
$\{(9^2 - 5^2)/2\}^2 + (9.5)^2 = \{(9^2 + 5^2)/2\}^2$
$\{(11^2 - 7^2)/2\}^2 + (11.7)^2 = \{(11^2 + 7^2)/2\}^2$
…………………………………………

For k = 18, $\{(11^2 - 5^2)/2\}^2 + (11.5)^2 = \{(11^2 + 5^2)/2\}^2$
$\{(13^2 - 7^2)/2\}^2 + (13.7)^2 = \{(13^2 + 7^2)/2\}^2$
$\{(15^2 - 9^2)/2\}^2 + (15.9)^2 = \{(15^2 + 9^2)/2\}^2$
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Note: Sum of squares of two odd integers d_1, d_2 is always in the form of 2d where d is a 2nd kind prime or purely 2nd kind composite when $(d_1, d_2) = 1$. For $(d_1, d_2) > 1$, d is a composite of 2nd kind nature or purely 2nd kind nature.

2. Two important operations

2.1 N_s-operation:

Identity for the product of two positive wings resulting equality of two positive wings as shown below, can be said as N_s-operation.

$(\alpha_1^2 + \beta_1^2)(\alpha_2^2 + \beta_2^2) = (\alpha_1\alpha_2 \pm \beta_1\beta_2)^2 + (\alpha_1\beta_2 \mp \alpha_2\beta_1)^2$

e.g. $65 = 5.13 = (2^2 + 1)(2^2 + 3^2) = (2.2 \pm 1.3)^2 + (2.3 \mp 2.1)^2 = 7^2 + 4^2 = 1^2 + 8^2$.

2.2 N_d-operation:

Identity for the product of two negative wings resulting equality of two negative wings as shown below, can be said as N_d-operation.

$(\alpha_1^2 - \beta_1^2)(\alpha_2^2 - \beta_2^2) = (\alpha_1\alpha_2 \pm \beta_1\beta_2)^2 + (\alpha_1\beta_2 \pm \alpha_2\beta_1)^2$

e.g. $35 = 5.7 = (3^2 - 2^2)(4^2 - 3^2) = (3.4 \pm 2.3)^2 - (3.3 \pm 2.4)^2 = 18^2 - 17^2 = 6^2 - 1^2$

3. How the elements of a N-equation $a^2 + b^2 = c^2$ produce power beyond two.

Here we consider 'a' as LH odd element, 'b' as LH even element & 'c' as RH odd element of the N-equation $a^2 + b^2 = c^2$ whose comparable equation is $(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2 = (\alpha^2 + \beta^2)^2$ where α, β can be said as mixed zygote elements & $(\alpha^2 \pm \beta^2)$ are the mixed zygote expression/wing of c or b conjugate to each other.

3.1 How the element 'a' produces power.

'a' produces power from 2 to 3 by virtue of N_d-operation in between $(\alpha^2 - \beta^2)$ & $(\alpha^2 - \beta^2)^2$
i.e. in between $(\alpha^2 - \beta^2)$ & $\{(\alpha^2 + \beta^2)^2 - (2\alpha\beta)^2\}$ & on multiplication we get,
$\{(\alpha^3 + \alpha\beta^2) \pm (2\alpha\beta^2)\}^2 - \{(2\alpha^2\beta) \pm (\alpha^2\beta + \beta^3)\}^2$
i.e. $(\alpha^3 + 3\alpha\beta^2)^2 - (3\alpha^2\beta + \beta^3)^2$ or $\{\alpha(\alpha^2 - \beta^2)\}^2 - \{\beta(\alpha^2 - \beta^2)\}^2$ where 2nd one can be neglected as it is a composite set.
$\Rightarrow (\alpha^2 - \beta^2)^3 = (\alpha^3 + 3\alpha\beta^2)^2 - (3\alpha^2\beta + \beta^3)^2$
By repeated multiplication of $(\alpha^2 - \beta^2)$ on both sides we get the general relation of 'a' in power form as
$(\alpha^2 - \beta^2)^n + \{{}^nc_1\alpha^{n-1}\beta + {}^nc_3\alpha^{n-3}\beta^3 + \ldots\ldots\}^2 = \{\alpha^n + {}^nc_2\alpha^{n-2}\beta^2 + \ldots\ldots\}^2$ where $n \in I$.

Note: for $n \in I_o$ it will always produce a relation like $a^{2n+1} + c^2 = b^2$ i.e. $(I_o)^{2n+1} + (I_o)^2 = (I_e)^2$ form, provided zygote expression of 'a' is in the form of $(I_e)^2 - (I_o)^2$ e.g. $3^3 + 13^2 = 14^2$ where $3 = 2^2 - 1^2$.

3.2 How the element 'c' produces power.

'c' produces power from 2 to 3 by virtue of N_s-operation in between $(\alpha^2 + \beta^2)$ & $(\alpha^2 + \beta^2)^2$
i.e. in between $(\alpha^2 + \beta^2)$ & $\{(\alpha^2 - \beta^2)^2 + (2\alpha\beta)^2\}$ & on multiplication we get,
$\{(2\alpha^2\beta) \pm (\alpha^2\beta - \beta^3)\}^2 + \{(\alpha^3 - \alpha\beta^2) \mp (2\alpha\beta^2)\}^2$
i.e. $(3\alpha^2\beta - \beta^3)^2 + (\alpha^3 - 3\alpha\beta^2)^2$ or $\{\alpha(\alpha^2 + \beta^2)\}^2 + \{\beta(\alpha^2 + \beta^2)\}^2$ where 2nd one can be neglected as it is a composite set. $\Rightarrow (\alpha^2 + \beta^2)^3 = (3\alpha^2\beta - \beta^3)^2 + (\alpha^3 - 3\alpha\beta^2)^2$
By repeated multiplication of $(\alpha^2 + \beta^2)$ on both sides we get the general relation of 'c' in power form as
$\{\alpha^n - {}^nc_2\alpha^{n-2}\beta^2 + \ldots\}^2 + \{{}^nc_1\alpha^{n-1}\beta - {}^nc_3\alpha^{n-3}\beta^3 + \ldots\}^2 = (\alpha^2 + \beta^2)^n$ where $n \in I$.

3.3 How the element 'b' produces power.

Before we analyze the power characteristic of element b, it is felt necessary to establish the following fundamental theorems.
For two odd integers x & y,

1. If $x, y \in$ 1st kind or 2nd kind then $xy \in$ 2nd kind.
2. If $x, y \in$ opposite kind then $xy \in$1st kind.
3. $\uparrow(x + y)_2 = 2$ when $x, y \in$ same kind and > 2 when $x, y \in$ opposite kind.
4. $\uparrow(x - y)_2 = 2$ when $x, y \in$ opposite kind and > 2 when $x, y \in$ same kind.
5. In a product of two odd negative wings $(c_1^2 - a_1^2)(c_2^2 - a_2^2) = (c_1c_2 + a_1a_2)^2 - (c_1a_2 + c_2a_1)^2$ if both $c \in$2nd kind & both $a \in$ same kind then $\uparrow(c_1c_2 + a_1a_2)_2 = 2 = \uparrow(c_1a_2 + c_2a_1)_2$

Here, b^2 is expressible in the form of $c^2 - a^2$ whereas b is expressible in the form of $(d_1^2 - d_2^2)/2$ where all elements are odd. So by N_d-operation we can't receive a relation like $b^3 = c^2 - a^2$ But for b^4 i.e. $b^2.b^2$ & for b^6 i.e. $b^4.b^2$ & b^8 i.e. $b^6.b^2$ & so on, we can always have a relation like $b^{2n} = c^2 - a^2$
Hence, $(\text{any even integer})^{\text{any odd integer}}$ cannot be a term of N-equation. It is under Nize-equation as per next theorem (Natural equation of irrational zygote elements)
So it is observed that 'a' produces power by N_d-operation among mixed zygote expressions i.e. mixed with odd & even elements.
'b' produces even power by N_d-operation among odd zygote expressions i.e. mixed with only odd elements.
'c' produces power by N_s-operation among mixed zygote expressions.
Any two of these three operations or all the three are not possible to be run simultaneously.
So in N-equation only one element can raise its power beyond two.
The general form of N-equation where even element 'b' is in power form by repeated applications of N_d-operations over $b^{2n-2}.b^2$, $n \in I$ can be written as,
$b^{2n} + ({}^nc_1c^{n-1}a + {}^nc_3c^{n-3}a^3 + \ldots)^2 = (c^n + {}^nc_2c^{n-2}a^2 + \ldots)^2$ which is a composite set with common factor 2^{n-1}
$\Rightarrow (b^n/2^{n-1})^2 + (d_1)^2 = (d_2)^2$ where obviously d_1 & d_2 are odd.
Say, $b = 2^m\alpha^p$ where α is odd. $\Rightarrow \{2^{n(m-1)+1}.\alpha^{pn}\}^2 + (d_1)^2 = (d_2)^2$, where GCF of $n(m - 1) + 1$ & $pn > 1$ so as to receive the even element in power form.
N-equation therefore, produces following three types of relations in power form.
$\pm a^{n+1} + b^2 = c^2$, $a^2 + b^2 = c^{n+1}$ & $a^2 + b^{2n} = c^2$ where $n \in I$ & (a, b, c) is a prime set.

4. How two elements are found to be in power form beyond two.

If the zygote elements are of irrational nature i.e. in the form of $(p \pm q\sqrt{r})$, we have the N-equation renamed as N-equation of irrational zygote elements or simply Nize-equation.
$\Rightarrow \{(p + q\sqrt{r})^2 - (p - q\sqrt{r})^2\}^2 + \{2(p + q\sqrt{r})(p - q\sqrt{r})\}^2 = \{(p + q\sqrt{r})^2 + (p - q\sqrt{r})^2\}^2$
Or, $(4pq\sqrt{r})^2 + \{2(p^2 - q^2r)\}^2 = \{2(p^2 + q^2r)\}^2 \Rightarrow \{(p)^2 - (q\sqrt{r})^2\}^2 + (2p.q\sqrt{r})^2 = \{(p)^2 + (q\sqrt{r})^2\}^2$

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Here also, like N-equation the RH term of Nize-equation can produce even power by virtue of N_s-operation in between the mother expression and self & odd power by same N_s-operation in between the mother expression and its irrational zygote expression.
Similar rule is applicable for LH term of Nize-equation by virtue of N_d-operation. But N_s & N_d both operations can't run simultaneously. Hence, elements under conjugate expressions $\alpha^2 \pm \beta^2$ can't raise its power beyond two simultaneously. But after N_s or N_d operation, the 3rd irrational element can produce power beyond two due to presence of $\sqrt{r}$ factor.
Here, obviously $(p, q) = (p, r) = 1$ & if $p \in I_e$, both $q, r \in I_o$ & if $p \in I_o$; q, r can be any combination.

Let us rewrite the N-equation in power form:
$(\alpha^2 - \beta^2)^n + \{{}^nc_1\alpha^{n-1}\beta + {}^nc_3\alpha^{n-3}\beta^3 + \ldots\ldots\}^2 = \{\alpha^n + {}^nc_2\alpha^{n-2}\beta^2 + \ldots\ldots\}^2$ ……… (A)
$\{\alpha^n - {}^nc_2\alpha^{n-2}\beta^2 + \ldots..\}^2 + \{{}^nc_1\alpha^{n-1}\beta - {}^nc_3\alpha^{n-3}\beta^3 + \ldots\ldots\}^2 = (\alpha^2 + \beta^2)^n$ ………. (B)
For Nize-equation where α is integer & β is irrational Eq. (A) can be written in two ways:
$(\alpha^2 - \beta^2)^n + \{\beta f(\alpha, \beta, n)\}^2 = \{\alpha g(\alpha, \beta, n)\}^2$ when n is odd ……. (A1)
$(\alpha^2 - \beta^2)^n + \{\alpha\beta f(\alpha, \beta, n)\}^2 = \{\alpha g(\alpha, \beta, n)\}^2$ when n is even ……. (A2)

The integer element i.e. 3rd one can't produce power. If the irrational element i.e. 2nd one produces power, $f(\alpha, \beta, n)$ must be in the form of β^{2m} in case of n is odd & in the form of $(\alpha\beta)^{2m}$ in case of n is even.
Similarly, Eq-B can be written in two ways:
$\{\alpha g_1(\alpha, \beta, n)\}^2 + \{\beta f_1(\alpha, \beta, n)\}^2 = (\alpha^2 + \beta^2)^n$ when n is odd ………. (B1)
$\{g_1(\alpha, \beta, n)\}^2 + \{\alpha\beta f_1(\alpha, \beta, n)\}^2 = (\alpha^2 + \beta^2)^n$ when n is even ………. (B2)

The integer element i.e. 1st one can't produce power. If the irrational element i.e. 2nd one produces power, $f_1(\alpha, \beta, n)$ must be in the form of β^{2m} in case of n is odd & in the form of $(\alpha\beta)^{2m}$ in case of n is even.

Examples in favor of Eq-(B1):

For $n = 3$, $3\alpha^2 - \beta^2 = \beta^m$ or, $\beta^m + \beta^2 - 3\alpha^2 = 0$ where obviously m is even & β is in the form of $q\sqrt{r}$ ($q, r \in I_o$) and $(\alpha, q) = 1 = (\alpha, r)$.
We have $(\sqrt{3})^4 + (\sqrt{3})^2 = 3.2^2$ and hence, consider the equation $\{2^2 - (\sqrt{3})^2\}^2 + (2.2\sqrt{3})^2 = \{2^2 + (\sqrt{3})^2\}^2$
i.e.$1^2 + (4\sqrt{3})^2 = 7^2$, Now by N_s-operation in between 7^2 & 7 i.e. in between $\{1^2 + (4\sqrt{3})^2\}$ & $\{2^2 + (\sqrt{3})^2\}$ we get,
$(8\sqrt{3} \pm \sqrt{3})^2 + (12 \mp 2)^2$ where one case is $3^5 + 10^2 = 7^3$

Examples in favor of Eq-(B2):

For $n = 4$, the irrational element is $\beta.4(\beta^2 - \alpha^2)$ where for $\alpha = 1$ & $\beta = \sqrt{2}$, $4(\beta^2 - \alpha^2) = 4 = (\sqrt{2})^4 = \beta^4$ Hence, consider the equation $\{(\sqrt{2})^2 - 1^2\}^2 + (2\sqrt{2})^2 = \{(\sqrt{2})^2 + 1^2\}^2$ or, $1^2 + (2\sqrt{2})^2 = 3^2$. Now by N_s-operation in between $\{(2\sqrt{2})^2 + 1^2\}$ & self we get $(2\sqrt{2} \pm 2\sqrt{2})^2 + (8 \mp 1)^2 = 3^2.3^2$ or, $2^5 + 7^2 = 3^4$
Or, directly from Eq-(B) we get the same result.
On the same logic for n= 2, we get $1 + 2^3 = 3^2$.

Note: As binomially expanded both the elements under Eq-(B) are sum of alternately (+) & (–), it will produce the relations of low value elements whereas Eq-(A) will produce relations of high value elements.
The above three examples are the particular cases of a general theory. By above concept it is not possible to extract many more examples from Eqs-(A), (B) & (C) because of following reasons.

For n is odd, $\beta f(\alpha, \beta, n) = \beta[{}^nc_1\alpha^{n-1} \pm X.\beta^2] = \beta.Y$ (say) form where the integer Y is free from α, β
$\Rightarrow \beta f(\alpha, \beta, n) \neq \beta^{\lambda/2}$ form where $\lambda \in I_o$
Due to same reason when n is even $\alpha\beta f(\alpha, \beta, n) = \alpha\beta[{}^nc_1\alpha^{n-2} \pm X.\beta^2] = \alpha\beta.Y$ form where the integer Y is free from α, β & $\Rightarrow \alpha\beta f(\alpha, \beta, n) \neq (\alpha\beta)^{\lambda/2}$ form where $\lambda \in I_o$

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Only possibility to extract more examples from the said equations is to follow a general theory.
Here, $\alpha \in I$ & $\beta \in$ irrational ($q\sqrt{r}$ form) where $(\alpha, q) = 1 = (\alpha, r)$
When n is odd $\alpha \in I$, but for n is even α must be equal to in the form of $p^{\lambda 1}$, $\lambda_1 \geq 3$ or 0 & in both cases β is in the form of $q^{\lambda 2}r^{\lambda 3/2}$ where $\lambda_2 \geq 3$ or 0 & $\lambda_3 \in I_o \geq 3$.
$\Rightarrow$ when $n \in I_0$, $b^2 = \{\beta f(\alpha, \beta, n)\}^2 = q^{2\lambda 2}r^{\lambda 3}\{f(\alpha, q^{\lambda 2}, r^{\lambda 3}, n)\}^2 = q^{2\lambda 2}r^{\lambda 3}.\omega^{2\lambda 4}$ (say, $\lambda_4 \in I \geq 3$) where $(\lambda_2, \lambda_3, \lambda_4) = \lambda$ & $\lambda \geq 3$
$\Rightarrow b^2 = b_1^s$ form. Similarly, when $n \in I_e$, $b^2 = p^{2\lambda 1}q^{2\lambda 2}r^{\lambda 3}\ \omega^{2\lambda 4}$ where same logic is applicable i.e. $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda$ & $\lambda \geq 3$
In view of the above, we can establish the following two conditions where two elements are in power form of a Nize-equation.

1. $|{}^nc_1p^{n-1} \pm {}^nc_3p^{n-3}\,q^{2\lambda 1}r^{\lambda 2} + {}^nc_5p^{n-5}q^{4\lambda 1}r^{2\lambda 2} \pm \ldots\ldots..| = \omega^{\lambda 3}$ where $(\lambda_1, \lambda_2, \lambda_3) \geq 3$ & $n \in I_o$
2. $|{}^nc_1\,p^{(n-2)\lambda 1} \pm {}^nc_3p^{(n-4)\lambda 1}\,q^{2\lambda 2}r^{\lambda 3} + {}^nc_5\,p^{(n-6)\lambda 1}\,q^{4\lambda 2}r^{2\lambda 3} \pm \ldots\ldots| = \omega^{\lambda 4}$ where $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \geq 3$ & $n \in I_e$

[Here, $r \neq 1$ but $p, q \geq 1$, $n \geq 3$, $\lambda_2 \in I_o$ in 1st case & $\lambda_3 \in I_o$ in the 2nd case, $(p, q) = 1 = (p, r)$]

Say, $n = 3 \Rightarrow |3p^2 \pm q^{2\lambda 1}r^{\lambda 2}| = \omega^{\lambda 3}$, $n = 5 \Rightarrow 5p^4 \pm 10p^2\,q^{2\lambda 1}r^{\lambda 2} + q^{4\lambda 1}r^{2\lambda 2} = \omega^{\lambda 3}$ & so on.
Say, $n = 4 \Rightarrow |4p^{2\lambda 1} \pm 4\,q^{2\lambda 2}r^{\lambda 3}| = \omega^{\lambda 4}$, $n = 6 \Rightarrow 6p^{4\lambda 1} \pm 20p^{2\lambda 1}q^{2\lambda 2}r^{\lambda 3} + 6q^{4\lambda 2}r^{2\lambda 3} = \omega^{\lambda 4}$ & so on.
If the said condition is satisfied we can have the relations like $a^n + b^{\lambda} = c^2$ or $a^2 + b^{\lambda} = c^n$, $(n, \lambda \geq 3)$

Examples in favor of Eq-(A1):

For n = 3, consider the relation $\{21^2 - (2\sqrt{2})^2\}^2 + (2.21.2\sqrt{2})^2 = \{21^2 + (2\sqrt{2})^2\}^2$ i.e. $433^2 + (84\sqrt{2})^2 = 449^2$
Now, applying N_d operation in between $449^2 - (84\sqrt{2})^2$ & $21^2 - (2\sqrt{2})^2$ we have,
$(449.21 \pm 336)^2 - \{84.21\sqrt{2} \pm 449.2\sqrt{2}\}^2 = 433^3$ & considering (+) sign, $433^3 + 242^3 = 9765^2$
This can be directly obtained from $(\alpha^2 - \beta^2)^3 + (3\alpha^2\beta + \beta^3)^2 = (\alpha^3 + 3\alpha^2\beta)^2$ where $\alpha = 21$ & $\beta = 2\sqrt{2}$
This type of Nize-relations are obtained from $(a^{2x+1} \pm b^{2x+1}) = I^2$.
We have, $(83^3 + 61^3)/3 = 516^2 \Rightarrow \{516^2 - (61\sqrt{61})^2\}^2 + (2.516.61\sqrt{61})^2 = \{516^2 + (61\sqrt{61})^2\}^2$
By N_s-operation on $\{516^2 + (61\sqrt{61})^2\}^2$ & $516^2 + (61\sqrt{61})^2$ i.e. $(39275)^2 + (62952\sqrt{61})^2$ & $516^2 + (61\sqrt{61})^2$
We get $213978492^2 + 420229^3 = 493237^3$
Again from $\{516^2 - (83\sqrt{83})^2\}^2 + (2.516.83\sqrt{83})^2 = \{516^2 + (83\sqrt{83})^2\}^2$ we get another relation like $747738180^2 + 308843^3 = 838043^3$.

Note: As N_d & N_s operations both cannot run simultaneously, exponent of one element must be restricted to 2 & this element is the integer part i.e. $(\alpha^3 + 3\alpha^2\beta)$ not $(3\alpha^2\beta + \beta^3)$ which is irrational part. If we assume that $(\alpha^3 + 3\alpha^2\beta)$ produces power then $(3\alpha^2\beta + \beta^3)$ becomes purely irrational i.e. a powerless term where exponent can be considered as 1 of an integer element which is neither N-eq. nor a Nize-eq. Hence, only irrational part can produce power beyond two for all cases.
Computer generated some examples are given below. All can be explained in the same way. Against each example there must exist a conjugate pair of irrational zygote expression, positive & negative and by successive N_s or N_d operations we can get the desired relation.

$7^3 + 13^2 = 2^9$;	$9262^3 + 15312283^2 = 113^7$;	$17^7 + 76271^3 = 21063928^2$;
$3^5 + 11^4 = 122^2$;	$43^8 + 96222^3 = 30042907^2$;	$33^8 + 1549034^2 = 15613^3$;
$17^3 + 2^7 = 71^2$;	$1414^3 + 2213459^2 = 65^7$; etc	

5. Beal Equation & Fermat's Last Theorem (FLT)

$a^x + b^y = c^z$ where x, y, z all greater than two, is known as Beal Equation & for a particular case when x = y = z it is called Fermat's Last Theorem (FLT).
Now it is quite obvious that if a, b, c are prime to each other a Beal equation or FLT cannot exist. If Beal equation exists there must be a common factor among a, b, c.

If we assume the existence of a Beal equation where (a, b, c) = 1, then particular two elements of a, b, c must be the zygote expressions conjugate to each other and by virtue of both N_d & N_s – simultaneous operations they are in power form which are quite impossible.
Hence, Beal equation or FLT cannot exist for (a, b, c) = 1

5.1 Theory behind the formation of Beal equation.

Algebraic sum of any two numbers both of which are in power form beyond two, is always in the form of $\lambda^\alpha\beta$ where $\alpha > 3$ & $\lambda \geq 1$ i.e. $A^m \pm B^n = \lambda^\alpha\beta$. Select a number p from 4, 5, 6, ……, α so that GCF of (p, α), (p – 1, m) & (p – 1, n) all are ≥ 3. If p exists then Beal equation will exist with a common multiplier β^{p-1}.
As the consecutive numbers p & (p – 1) don't have any common factor in between them, hence for any Beal equation $a^x + b^y = c^z$, (x, y, z) is a prime set i.e. no common factor lies among x, y, z
In between any two there can be a common factor. This implies, all the powers cannot be even.
As $\alpha \leq \min(m, n)$ when $\lambda \neq 1$, it is quite obvious that to produce Beal equation m, n both must be composite & α may be composite or may be prime. For $\lambda = 1$, m & n both can be prime & in this case α must be even i.e. composite.
Among all the bases a, b, c there must be a common factor without which a Beal equation can't exist.

5.2 Few examples for the existence of $A^m \pm B^n = \lambda^\alpha\beta$ where $\alpha > 3$ & $\lambda \geq 1$

5.2.1 We have 1 + 1 = 2 i.e. in the form of $1^m + 1^n = 1^\alpha.2$ Multiply both sides by $2^{\alpha-1}$ to get an example set of Beal equation $2^{\alpha-1} + 2^{\alpha-1} = 2^\alpha$ where α = 4, 5, 6, ……

5.2.2 Say, there lies a common factor in between A & B i.e. A = aθ & B = bθ & m > n
$\Rightarrow A^m \pm B^n = \theta^{m-n}.\beta$ Multiply both sides by β^{m-n-1} to get a relation
$A^m \beta^{m-n-1} \pm B^n \beta^{m-n-1} = (\beta\theta)^{m-n}$ from where infinite examples of Beal equation can be extracted satisfying the conditions (m, m – n – 1) ≥ 3 & (n, m – n – 1) ≥ 3

5.2.3 We have always $A^m \pm B^n = 1^\alpha.\beta$ & choose α so that (m, α – 1) = x & (n, α – 1) = y where x, y > 2 to get the infinite examples of Beal equation $a^x + b^y = \beta^\alpha$.

5.2.4 Say, A, B, m, n all $\in I_o$ for $N = A^m - B^n = (1 + e_1)^m - (1 + e_2)^n$ where $\uparrow(e_1)_2 = p$ & $\uparrow(e_2)_2 = q$ & p > q > 3.
After binomial expansion we have $N = e_1(I_o^{/}) - e_2(I_o^{//}) \Rightarrow \uparrow(N)_2 = q$ for p ≠ q and > q for p = q. In both cases $N = 2^\alpha.\beta$ form i.e. $A^m - B^n = 2^\alpha.\beta$ where $\alpha > 3$. This is also true for m, n both are even or combination of even & odd. But for m, n $\in I_e$, p > q > 2
For $\uparrow(e_1)_x = p$ & $\uparrow(e_2)_x = q$ & p > q > 3 where x is a common prime factor same is applicable to form a Beal eq. for any combination of A & B. But so far $N = A^m + B^n$ is concerned at least one of m, n must be odd.

5.2.5 For any combination of A, B say (m, n) = p^α where p $\in$ prime & $\alpha \geq 4$
$\Rightarrow N = A^m - B^n = \{(1 + x)^p\}^{\lambda 1} - \{(1 + y)\}^{\mu 1} = \{(1 + px_1)\}^{\lambda 1} - \{(1 + py_1)\}^{\mu 1} = \{(1 + px_1)^p\}^{\lambda 2} - \{(1 + py_1)^p\}^{\mu 2}$
$= \{(1 + p^2x_2)\}^{\lambda 2} - \{(1 + p^2y_2)\}^{\mu 2} = \{(1 + p^2x_2)^p\}^{\lambda 3} - \{(1 + p^2y_2)^p\}^{\mu 3} = \{(1 + p^3x_3)\}^{\lambda 3} - \{(1 + p^3y_3)\}^{\mu 3} = \ldots\ldots\ldots$
Finally, $N = (1 + p^\alpha u)^\lambda - (1 + p^\alpha v)^\mu = p^\alpha.\beta$ form.
It is also true for $N = A^m + B^n$ where m, n both cannot be even.
A ± 1 & B ± 1 should be chosen in such a way so that in binomial expansion one is cancelled out.

5.2.6 $N = A^{2m} + B^{2n}$ fails to produce a Beal equation by virtue of any common factor in between (A ± 1, B ± 1) or (m, n). Here, $N = 1^\alpha(2I_o)$ form. It can produce a Beal equation by virtue of (2m, α – 1) ≥ 3 and (2n, α – 1) ≥ 3. Beal equation produced by considering the factor 1^α in all cases , can be said as Auto-Beal equation.

6. Some important corollaries:

6.1 For all prime numbers ‘P’

Digit of unit place of P	Digit of 10th place of P	Nature of P	Remarks
1 or 9	Even	2nd kind prime(P_2)	$\uparrow(P_2 - 1)_2 > 1$
3 or 7	Odd	2nd kind prime(P_2)	$\uparrow(P_2 - 1)_2 > 1$
1 or 9	Odd	1st kind prime(P_1)	$\uparrow(P_1 - 1)_2 = 1$
3 or 7	Even	1st kind prime(P_1)	$\uparrow(P_1 - 1)_2 = 1$

6.2 Any 2nd kind prime irrespective of its exponent has a single positive prime wing and a single negative prime wing with consecutive elements. Elements of the wing are different for different values of exponent. All the composite numbers that are capable of forming the positive wings contain the prime factors of 2nd kind only. All 1st kind primes irrespective of its exponent have a single negative prime wing, elements of which are consecutive but go on changing as exponent changes. Hence, total number of prime wings (positive or negative) of a composite number depends upon the nos. of prime factors it contains, not the exponents of individual prime factor. If the number of prime factors is n total number of prime wings (positive or negative) produced is 2^{n-1} with different wing lengths or elementary gap and where all the produced elements (2^n nos.) are different.

$\prod_{i=1}^{n}(P_i^{n_i}) = P_1^{n_1} P_2^{n_2} P_3^{n_3}$…….. n terms (all $P_i \in$ 2nd kind prime)

$= \prod(e_i^2 + o_i^2) = (e_1^2 + o_1^2)(e_2^2 + o_2^2)(e_3^2 + o_3^2)\ldots\ldots\ldots (e_n^2 + o_n^2)$

$= E(v_i^2 + d_i^2) = (v_1^2 + d_1^2) = (v_2^2 + d_2^2) = (v_3^2 + d_3^2) = \ldots\ldots\ldots$ 2^{n-1} prime wings.

Similarly, for $P_i \in$ 1st kind or 2nd kind prime & for negative prime wings

$\prod(P_i^{n_i}) = \prod(e_i^2 \sim o_i^2) = (e_1^2 \sim o_1^2)(e_2^2 \sim o_2^2)(e_3^2 \sim o_3^2)\ldots\ldots\ldots (e_n^2 \sim o_n^2)$

$= E(v_i^2 \sim d_i^2) = (v_1^2 \sim d_1^2) = (v_2^2 \sim d_2^2) = (v_3^2 \sim d_3^2) = \ldots\ldots\ldots$ 2^{n-1} prime wings.

$[v_i \neq v_{i+j}$ & $d_i \neq d_{i+j}$, e & v $\in I_e$, o & d $\in I_o]$

6.3 In a N-equation, RH odd element ± LH odd element = 2(integer)^2
& RH odd element ± LH even element = (odd integer)^2

6.4 All 2nd kind primes are distributed to satisfy ‘c’ of a N-equation $a^2 + b^2 = c^2$ for all different values of ‘k’ whereas all 1st kind & 2nd kind primes are confined to k = 1 only to satisfy ‘a’ of a N-equation.

6.5 If $(a_1^2 - b_1^2)(a_2^2 - b_2^2)$ produces a relation $(a_3^2 - b_3^2) = (a_4^2 - b_4^2)$, then $(a_1^2 + b_1^2)(a_2^2 + b_2^2)$ will produce a relation $(a_3^2 + b_4^2) = (a_4^2 + b_3^2)$.$(a_1^2 \pm b_1^2)$, $(a_2^2 \pm b_2^2)$ can be said as product wings and all the wings under equality can be said as produced wings.

6.6 If $(e_1^2 + o_1^2)(e_2^2 + o_2^2)$ produces a relation $(e_3^2 + o_3^2) = (e_4^2 + o_4^2)$ where e, o denote even & odd respectively then $\mathrm{Max}(e_3, e_4) + \mathrm{Max}(o_3, o_4) = (e_1 + o_1)(e_2 + o_2)$ i.e. a composite number &
$|\mathrm{Max}(e_3, e_4) - \mathrm{Max}(o_3, o_4)| = |(e_1 - o_1)(e_2 - o_2)|$ which may be prime or composite.

6.7 The product & division rules of N_s-operation are as follows:

$(e_1^2 + o_1^2).(e_2^2 + o_2^2) = (|e_1e_2 \pm o_1o_2|)^2 + (|e_1o_2 \mp o_1e_2|)^2$ and

$(e_1^2 + o_1^2)/(e_2^2 + o_2^2) = \{|e_1e_2 \pm o_1o_2|/(e_2^2 + o_2^2)\}^2 + \{|e_1o_2 \mp o_1e_2|/(e_2^2 + o_2^2)\}^2$ consider only one wing which has integer elements.

6.8 The product & division rules of N_d-operation are as follows:

$(e_1^2 - o_1^2).(e_2^2 - o_2^2) = (|e_1e_2 \pm o_1o_2|)^2 + (|e_1o_2 \pm o_1e_2|)^2$ and

$(e_1^2 - o_1^2)/(e_2^2 - o_2^2) = \{|e_1e_2 \pm o_1o_2|/(e_2^2 - o_2^2)\}^2 + \{|e_1o_2 \pm o_1e_2|/(e_2^2 - o_2^2)\}^2$ consider only one wing which has integer elements.

6.9 For both kind N-equations $a^2 + b^2 = c^2$, c cannot be expressed as product of two algebraic linear factors. It can be easily proved by the functional form of c i.e. $2y^2 + 2y(2x - 1) + (2x - 1)^2$ for 1st kind and $y^2 + 2xy + 2x^2$ for 2nd kind where x is a fixed variable. In both cases D < 0 for the quadratic equation of y

6.8 In a mixed combination of x, y; $\{x^n + {}^nc_2x^{n-2}y^2 + {}^nc_4x^{n-4}y^4 + \ldots\ldots\}$ always represents a 2nd kind prime or a pure 2nd kind composite i.e. capable of producing a positive prime wing for (x, y) = 1

6.10 $F(x, y) = \{x^n \pm {}^nc_2x^{n-2}y^2 + {}^nc_4x^{n-4}y^4 \pm \ldots\}$ & $G(x, y) = \{{}^nc_1x^{n-1}y \pm {}^nc_3x^{n-3}y^3 + {}^nc_5x^{n-5}y^5 \pm \ldots\}$ both are polynomials of power-free integers for $n > 2 \Rightarrow F\{f(x), g(y)\}$ or $G\{f(x), g(y)\}$ where f(x), g(y) are polynomials of integer coefficients, always represent a power-free integer.

6.11 For k = 1 & 2 in a N-equation $a^2 + b^2 = c^2$, the ordered pair (b ~ a, a + b) follows the sequence (α_1, α_2), (α_2, α_3), (α_3, α_4), (α_4, α_5),………….
For k = 1, the functional form of $a(x, y) = 2y(2x - 1) + (2x - 1)^2$ & $b(x, y) = 2y(y + 2x - 1)$ where x is a fixed variable. $\Rightarrow b(x, y) - a(x, y) = 2y^2 - (2x - 1)^2$ & $b + a = 2y^2 + 4y(2x - 1) + (2x - 1)^2$
Now, equating $b(x, y + 1) - a(x, y + 1) = b(x, y) + a(x, y)$ we get $(y + x)(x - 1) = 0 \Rightarrow x = 1 \Rightarrow k = 1$

Similarly, considering the functional form of 2nd kind N-equation we can get the same sequence for x = 1 i.e. k = 2. Here, $b(x, y) = (2x - 1)^2 + 2y(2x - 1)$ & $a(x, y) = 2y^2 + 2y(2x - 1)$ where y is fixed variable and equating $b(x + 1, y) - a(x + 1, y) = b(x, y) + a(x, y)$ we get $2(4x + y)(y - 1) = 0$
$\Rightarrow y = 1$ i.e. k = 2.
For k = 1, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $7^2 + 24^2 = 25^2$, $9^2 + 40^2 = 41^2$ ………
$\Rightarrow (b - a, b + a) \equiv (1, 7), (7, 17), (17, 31), (31, 49)$ ……….
For k = 2, $8^2 + 15^2 = 17^2$, $12^2 + 35^2 = 37^2$, $16^2 + 63^2 = 65^2$, $20^2 + 99^2 = 101^2$ ………
$\Rightarrow (b - a, b + a) \equiv (7, 23), (23, 47), (47, 79), (79, 119)$ ……….

7.1 How a, b form consecutive integers of a N-equation $a^2 + b^2 = c^2$

If a is odd & b is even then from property of N-equation we can write c + a & c + b are of the form $2\beta^2$ & α^2 where $\alpha \in I_o \Rightarrow a \sim b$ is of the form $\alpha^2 \sim 2\beta^2 \Rightarrow$ if a ~ b = 1 then $\alpha^2 \sim 2\beta^2 = 1$

Case I: When $\alpha^2 - 2\beta^2 = 1$.
Say, $\alpha = (2x + 1) \Rightarrow \beta^2 = 2x(x + 1)$ which is possible only for x = 1 i.e. $3^2 - 2.2^2 = 1$
It produces the only relation under 2nd kind N-equation i.e. $20^2 + 21^2 = 29^2$

Case II: When $2\beta^2 - \alpha^2 = 1$.
Here $\beta^2 = x^2 + (x + 1)^2 \Rightarrow$ For $k = (2x + 1)^2$ there must exist a relation $x^2 + (x + 1)^2 = \beta^2$
We have $3^2 + 4^2 = 5^2$ where $x = 3 \Rightarrow$ next consecutive phenomenon will be observed for $(2.3 + 1)^2 < 50$ i.e. 2.5^2 i.e. for $k = 7^2$.
From $(b - 1)^2 + b^2 = (b + 7^2)^2$ we get b = 120 that follows the relation $119^2 + 120^2 = 169^2$.
Next consecutive phenomenon will be observed for $(2.119 + 1)^2 < 2.169^2$ i.e. for $k = 239^2$.
Again from $(b - 1)^2 + b^2 = (b + 239^2)^2$ we get b = 137904 that implies $137903^2 + 137904^2 = 195025^2$.
Similarly, next ab-consecutive phenomenon will be observed as $183648021599^2 + 183648021600^2 = 259717522849^2$ and so on.
So, with the help of an ab-consecutive N-equation (or can be said as abc-eq.) we can produce next abc-equation and hence its existence is infinitely extended.

abc-equation always falls under 1st kind except $20^2 + 21^2 = 29^2$ which is under 2nd kind. This is because of the fact that $2u^2 - 1 = I^2$ has infinitely many solutions but $2u^2 + 1 = I^2$ has the only solution i.e. $2.2^2 + 1 = 3^2$.
If abc-eq. falls under $k = p^2$ then $p^2 + 1$ must be in the form of $2u^2$.
Obviously, abc-equation is the leading set of $k = p^2$.

7.2 For a N-equation $a^2 + b^2 = c^2$ (a < b), c^4 will produce an abc-equation having $k = (b - a)^2$ if $(b - a)^2$ is of the form $2u^2 - 1$.

$a^2 + b^2 = c^2$ may be of any kind and accordingly $(b^2 - a^2)^2 + (2ab)^2 = c^4$ may represent any kind of N- equation but c^4 will produce abc-equation when it is of 1st kind i.e. $(b^2 - a^2) < (2ab)$
Let's consider a N-eq. of 1st kind $\{b + (2x - 1)^2 - 2y^2\}^2 + b^2 = \{b + (2x - 1)^2\}^2 = c^2$ where $2y^2 > (2x - 1)^2$
$\Rightarrow [b^2 - \{b + (2x - 1)^2 - 2y^2\}^2]^2 + [2b\{b + (2x - 1)^2 - 2y^2\}]^2 = c^4$
$\Rightarrow 2b\{b + (2x - 1)^2 - 2y^2\} - [b^2 - \{b + (2x - 1)^2 - 2y^2\}^2] = 1$
$\Rightarrow 2b^2 - 4bp + p^2 - 1 = 0$ where $p = 2y^2 - (2x - 1)^2$ & $D = 8(p^2 + 1)$
Obviously, if D is a square integer p^2 i.e. $(b - a)^2$ must be in the form of $2u^2 - 1$.
Similarly, for 2nd kind N-eq. also we will get the same result i.e. $(b - a)^2$ is in the form of $2u^2 - 1$.

7.3 There exists only one relation under k = 1 where c^4 produces abc-equation.

Functional form of N-equation for k = 1 is $(2x + 1)^2 + \{2x(x + 1)\}^2 = c^2$
$\Rightarrow [\{2x(x + 1)\}^2 - (2x + 1)^2]^2 + \{2.2x(x + 1)(2x + 1)\}^2 = c^4$ where
$\{2.2x(x + 1)(2x + 1)\} - [\{2x(x + 1)\}^2 - (2x + 1)^2] = 1$ & on simplification $x^3 - 3x - 2 = 0$ i.e. $x = 2$
$\Rightarrow 5^2 + 12^2 = 13^2$ produces abc-eq. on squaring i.e. $(12^2 - 5^2)^2 + (2.12.5)^2 = 13^4$ i.e. $119^2 + 120^2 = 13^4$

7.4 For a sequence of N-equation $(a_i)^2 + (b_i)^2 = (c_i)^2$ of a particular value of k, c^4 will remain under 1st kind so long $2a^2 - (b - a)^2 > 0$. If $2a^2 - (b - a)^2 = 1$, c^4 will produce abc-relation where c^2 can be said as square root of abc-equation or simply √abc-equation.

Let us consider a N-equation of any kind $a^2 + b^2 = c^2$ where b > a
$\Rightarrow (b^2 - a^2)^2 + (2ab)^2 = c^4$ will remain under 1st kind when $2ab > (b^2 - a^2)$ i.e. $(b/a)^2 - 2(b/a) - 1 < 0$
$\Rightarrow b/a \in (0, \sqrt{2} + 1)$. As $b/a > 1$, $b/a \in (1, \sqrt{2} + 1) \Rightarrow b/a < \sqrt{2} + 1 \Rightarrow 2a^2 - (b - a)^2 > 0$
Now, say $2a^2 - (b - a)^2 = 1$ i.e. $a^2 + 2ab - b^2 - 1 = 0$
Considering it as a quadratic equation of a, $a = - b + \sqrt{(2b^2 + 1)}$ where $(2b^2 + 1)$ is a square integer only for b = 2
$\Rightarrow$ for b = 2, a = 1 & √abc-equation is $1^2 + 2^2 = 5$ against abc-eq. $(2^2 - 1^2)^2 + (2.1.2)^2 = 5^2$ i.e. $3^2 + 4^2 = 5^2$. It cannot be continued as $(2b^2 + 1)$ fails to produce further square integer.
But considering the quadratic equation with respect to b we have $b = a + \sqrt{(2a^2 - 1)}$ where there exists infinite nos. of square integers against $(2a^2 - 1)$. First one is obviously 5 & for a = 5, b = 12 i.e. $5^2 + 12^2 = 13^2$ is the √abc-eq. of abc-eq. $(12^2 - 5^2)^2 + (2.5.12)^2 = (13^2)^2$ i.e. $119^2 + 120^2 = 169^2$
Next square integer of $(2a^2 - 1)$ is for a = 169 & for a = 169, b = 169 + 239 = 408
$\Rightarrow 169^2 + 408^2 = 195025$ is the √abc-eq. of abc-eq. $(408^2 - 169^2)^2 + (2.408.169)^2 = 195025^2$
i.e. $137903^2 + 137904^2 = 195025^2$.
Next a = 195025 & b = 195025 + 275807 = 470832
$\Rightarrow 195025^2 + 470832^2 = 259717522849$ is the √abc-eq. of abc-eq. $(470832^2 - 195025^2)^2 + (2.470832.195025)^2 = 259717522849^2$ i.e. $183648021599^2 + 183648021600^2 = 259717522849^2$ & so on.
Sequence of √abc-equations is given below:
$5^2 + \{5 + \sqrt{(2.5^2 - 1)}\}^2$ i.e. $5^2 + 12^2 = 13^2 = 169$
$169^2 + \{169 + \sqrt{(2.169^2 - 1)}\}^2$ i.e. $169^2 + 408^2 = 195025$
$195025^2 + \{195025 + \sqrt{(2.195025^2 - 1)}\}^2$ i.e. $195025^2 + 470832^2 = 259717522849$ & so on.
It is observed that every abc-relation has a definite √abc-relation which is obtained considering the sequence $2a^2 - (b - a)^2 = 1$

7.5 For a particular value of k of a N-equation there can exist only one √abc-relation.

For a 1st kind N-equation $a^2 + b^2 = c^2$ say, $k = (2y - 1)^2 = \alpha^2$
$\Rightarrow a^2 + b^2 = (\alpha^2 + 2x\alpha)^2 + (2x^2 + 2x\alpha)^2$ in functional form where x is variable & α is fixed variable.
$\Rightarrow$ applying condition of abc-equation, $2(\alpha^2 + 2x\alpha)^2 - (2x^2 - \alpha^2)^2 = 1$
$\Rightarrow 4x^4 - (12\alpha^2)x^2 - (8\alpha^3)x - (\alpha^4 - 1) = 0$ …….. (1)
Similarly for 2nd kind N-eq. having $k = 2\alpha^2$, $a^2 + b^2 = \{2\alpha^2 + 2\alpha(2y - 1)\}^2 + \{(2y - 1)^2 + 2\alpha(2y - 1)\}^2 = (2\alpha^2 + 2\alpha x)^2 + (x^2 + 2\alpha x)^2$ where α is fixed variable & x is variable.
$\Rightarrow$ Applying condition of abc-equation, $2(2\alpha^2 + 2\alpha x)^2 - (x^2 - 2\alpha^2)^2 = 1$
$\Rightarrow x^4 - (4\alpha^2)x^2 - (16\alpha^3)x - (4\alpha^4 - 1) = 0$……… (2)
Both the equations (1) & (2) confirm single positive root for a particular value of α. Hence, it is proved.
e.g. for (1) it is quite understood that it is satisfied by $\alpha = 1$ for which $x^3 - 3x - 2 = 0$ i.e. $x = 2$
$\Rightarrow 5^2 + 12^2 = 13^2$ and squaring both sides we get $119^2 + 120^2 = 169^2$

7.6 Equality of wing lengths with respect to N-equation.
7.6.1 For opposite kind of N-equation.

Let's consider two opposite kind N-equations $a^2 + b^2 = c^2$ $(a < b < c)$ having $k = (2\alpha - 1)^2$ & $k = 2\beta^2$
Equating their wing lengths (b – a) we have, $2x^2 - (2\alpha - 1)^2 = (2y - 1)^2 - 2\beta^2$ for $2x^2 > (2\alpha - 1)^2$ & $(2y - 1)^2 > 2\beta^2$
$\Rightarrow 2(x^2 + \beta^2) = (2\alpha - 1)^2 + (2y - 1)^2$
Obviously, equality holds good for odd-even mixed combination of x & β.
Such type of relation we receive while equating the mixed zygote form & odd zygote form of 'c' of a N-equation (Ref: 1.5.2 & 1.6.1)
Example: $5 = (3^2 + 1^2)/2 = 2^2 + 1^2$ where one case is $2.2^2 - 1^2 = 3^2 - 2.1^2$
Comparing it with functional form we have, $2x^2 - (2.1 - 1)^2 = (2y + 1)^2 - 2.1^2$
[it is (2y + 1) not (2y – 1) as because for y = 1 it must be greater than 2.1^2]
$\Rightarrow$ WL of xth relation (for k = 1) = WL of yth relation (for $k = 2.1^2$)
i.e. wing-length of 2nd relation (for k = 1) = wing-length of 1st relation (for $k = 2.1^2$)
$\Rightarrow$ WL of $(5^2 + 12^2 = 13^2)$ = WL of $(8^2 + 15^2 = 17^2)$. In both cases WL is 7
Again, $2.1^2 - 1^2 = 3^2 - 2.2^2 \Rightarrow 2x^2 - 1^2 = (2y + 1)^2 - 2.2^2$
$\Rightarrow$ WL of 1st relation (for k = 1) = WL of 1st relation (for $k = 2.2^2$)
$\Rightarrow$ WL of $(3^2 + 4^2)$ = WL of $(20^2 + 21^2)$. In both cases it is 1.
$13 = (5^2 + 1^2)/2 = 3^2 + 2^2 \Rightarrow 2.3^2 - 1^2 = 5^2 - 2.2^2$ i.e. $2x^2 - 1^2 = (2y + 1)^2 - 2.2^2$
$\Rightarrow$ wl of 3rd relation of (k = 1) i.e. $7^2 + 24^2$ = wl of 2nd relation of ($k = 2.2^2$) i.e. $28^2 + 45^2$, in both cases it is 17.
$65 = 8^2 + 1^2 = 7^2 + 4^2 = (11^2 + 3^2)/2 = (9^2 + 7^2)/2$ where one case is $2.7^2 - 3^2 = 11^2 - 2.4^2$
$\Rightarrow 2(x + 2)^2 - 3^2 = (2y + 5)^2 - 2.4^2 \Rightarrow$ WL of 5th relation (of $k = 3^2$) = WL of 3rd relation (of $k = 2.4^2$)
$\Rightarrow$ WL of $(51^2 + 140^2 = 149^2)$ = WL of $(51^2 + 140^2 = 149^2)$. In both cases it is 89 & so on.

7.6.2 For same kind of N-equation.

Equality of wing lengths from 1st kind N-eq. means $2x^2 - (2\alpha - 1)^2 = 2y^2 - (2\beta - 1)^2$ where $2x^2 > (2\alpha - 1)^2$ & $2y^2 > (2\beta - 1)^2 \Rightarrow 2(x^2 - y^2) = (2\alpha - 1)^2 - (2\beta - 1)^2$ & 2nd kind means $(2x - 1)^2 - 2\alpha^2 = (2y - 1)^2 - 2\beta^2$ where $(2x - 1)^2 > 2\alpha^2$ & $(2y - 1)^2 > 2\beta^2$. Obviously equality holds good for x, y both are odd or both are even.
When both are odd such type of relation we receive while equating the odd zygote form of even element b & 2b of a N-equation.
e.g. $4 = (3^2 - 1^2)/2 \Rightarrow ½.8 = (3^2 - 1^2)/2 \Rightarrow ½.(5^2 - 3^2)/2 = (3^2 - 1^2)/2 \Rightarrow 5^2 - 3^2 = 2(3^2 - 1^2)$ but it fails to be arranged as per $2x^2 - (2\alpha - 1)^2 = 2y^2 - (2\beta - 1)^2$ form. $\Rightarrow 5^2 - 2.3^2 = 3^2 - 2.1^2$
$\Rightarrow (2x + 3)^2 - 2.3^2 = (2y + 1)^2 - 2.1^2$
$\Rightarrow$ WL of 1st relation under ($k = 2.3^2$) = WL of 1st relation under ($k = 2.1^2$)

$\Rightarrow$ WL of $(48^2 + 55^2 = 73^2)$ = WL of $(8^2 + 15^2 = 17^2)$. In both cases it is 7

Similarly, $8 = (5^2 - 3^2)/2 = 16/2 = ½.(9^2 - 7^2)/2 \Rightarrow 2(5^2 - 3^2) = 9^2 - 7^2$ which also fails to be arranged as per desired form $2x^2 - (2\alpha - 1)^2 = 2y^2 - (2\beta - 1)^2$ form. $\Rightarrow 9^2 - 2.5^2 = 7^2 - 2.3^2 \Rightarrow (2x + 7)^2 - 2.5^2 = (2y + 3)^2 - 2.3^2$

$\Rightarrow$ WL of 1st relation under $(k = 2.5^2)$ = WL of 2nd relation under $(k = 2.3^2)$

$\Rightarrow$ WL of $(140^2 + 171^2 = 221^2)$ = WL of $(60^2 + 91^2 = 109^2)$. In both cases it is 31.

So, equating the even element b of a N-equation for k = 1, with ½(2b) by odd-zygote form where obviously 2b satisfies N-eq. not for k = 1, we will always receive such type of failure cases.

i.e. $2(d_1^2 - d_2^2) \neq d_3^2 - d_4^2$ when $d_2 = 1$ for k = 1 in view of equal wing-length under 1st kind.

So, equality of wing lengths for non-square integer (as per next theorem) in between $k = d_1^2$ & $k = d_2^2$ of a 1st kind N-equation hold good either for $2(2x)^2 - d_1^2 = 2(2y)^2 - d_2^2$ …… (1) no. or,

$2(2x + 1)^2 - d_1^2 = 2(2y + 1)^2 - d_2^2$ …. (2) no. for $d_1, d_2 > 1$ where WL = I_0^2 shown in the next theorem

Examples: $2.2^2 - 1^2 = 2.4^2 - 5^2$ or, $2x^2 - 1^2 = 2y^2 - 5^2$

$\Rightarrow$ WL of 2nd relation under $(k = 1^2)$ = WL of 1st relation under $(k = 5^2)$

$\Rightarrow$ WL of $(5^2 + 12^2 = 13^2)$ = WL of $(55^2 + 62^2 = 97^2) = 7$

Again, $2.4^2 - 3^2 = 2.6^2 - 7^2$ or, $2(x + 2)^2 - 3^2 = 2(y + 4)^2 - 7^2$

$\Rightarrow$ WL of 2nd relation under $(k = 3^2)$ = WL of 2nd relation under $(k = 7^2)$

$\Rightarrow$ WL of $(33^2 + 56^2 = 65^2)$ = WL of $(133^2 + 156^2 = 205^2) = 23$ & so on.

7.7.3 Equality of wing-length which is a square integer always takes place in between k = 1 & k ≠ 1for 1st kind N-eq.

All though there is no existence for eq. 2 in the previous discussion 7.7.2 (proof given later) but we can have the matching form in the following way.

So far unit wing-length is concerned, it happens for $k = 1^2, 7^2, 239^2, 275807^2$,…….. as because

$1 = 2.1^2 - 1 = 2.5^2 - 7^2 = 2.169^2 - 239^2 = \ldots\ldots$ that help finding the sequence in equality of wing-length equals to a square integer. This nature of k can symbolically denoted by $k \in I_0(1, n^2)$

Functional form of equality for k = 1 & $k = d^2$ can be written as,

$2x^2 - 1 = 2(y + p)^2 - d^2$ where $2(y + p)^2 > d^2$, $d \in I_0 > 1$

i.e. WL of xth relation for (k = 1) = WL for a particular relation of $(k = d^2)$

From 2nd relation i.e. $1 = 2.5^2 - 7^2$ we can write $2.5^2 - 1 = 2.7^2 - 7^2$ or, $2.5^2 - 1 = 2.(y + 4)^2 - 7^2$

$\Rightarrow$ WL of 5th relation (of k = 1) = WL of 3rd relation (of $k = 7^2$)

$\Rightarrow$ WL of $(11^2 + 60^2 = 61^2)$ = WL of $(147^2 + 196^2 = 245^2)$ In both cases it is 49.

From 3rd relation i.e. $1 = 2.169^2 - 239^2$ we can write $2.169^2 - 1 = 2.239^2 - 239^2$ or, $2.169^2 - 1 = 2.(y + 168)^2 - 239^2$

$\Rightarrow$ WL of 169th relation (of k = 1) = WL of 71st relation (of $k = 239^2$)

Similarly from 4th relation i.e. $1 = 2.195025^2 - 275807^2$ we have,

$\Rightarrow$ WL of 195025th relation (of k = 1) = WL of 80783rd relation (of $k = 275807^2$) & so on.

For any kind N-eq. $a^2 + b^2 = c^2$, c^2 is always in the form of $c^2 = (d_1^2 + d_2^2)/2 \Rightarrow 2c^2 - d_1^2 = 2d_2^2 - d_2^2$

or, $2c^2 - d_2^2 = 2d_1^2 - d_1^2 \Rightarrow w(x) = d_1^2, d_2^2$ for both $k = d_1^2$ & $k = d_2^2$ for some $x = x_i$

This type of k in pairs can be denoted by $k \in I_0(d_i^2, d_j^2)$ where $d_i, d_j > 1$.

Examples: $17^2 = (23^2 + 7^2)/2 \Rightarrow 2.17^2 - 7^2 = 2.23^2 - 23^2 \Rightarrow 2(x + 4)^2 - 7^2 = 2(y + 16)^2 - 23^2$

$\Rightarrow$ WL of 13th relation of $(k = 7^2)$ = WL of 7th relation of $(k = 23^2)$

Whenever it is found that $w(x) = d_1^2$, there must exist another $w(x) = d_2^2$ so that both w(x) for $k = d_1^2$ & $k = d_2^2$ meet at d_1^2, d_2^2 for some values of x. One is prime wing & other is composite wing.

8. Some important theorems

8.1 The relation $2x^2 = d_1^2 + d_2^2$ exists when $x \in I_0$ and has no existence when $x \in I_e$ for $d_i > 1$.

$2x^2 = d_1^2 + d_2^2 = x^2 - d_1^2 = d_2^2 - x^2$ & squaring both sides $(x^2 - d_1^2)^2 = d_2^4 + x^4 - 2(xd_2)^2$

$\Rightarrow 2(xd_2)^2 - d_2^4 = x^4 - (x^2 - d_1^2)^2$ or, $d_2^2(2x^2 - d_2^2) = (x^2)^2 - (x^2 - d_1^2)^2$
$\Rightarrow d_2^2$(WL of $k = d_2^2$) $= (x^2)^2 - (x^2 - d_1^2)^2 \Rightarrow (x^2)^2 - (x^2 - d_1^2)^2 \in I_o^2$ Obviously, $x^2 \in I_0$ & $x^2 \in c^2$.

8.2 $2d^2 = d_1^2 - d_2^2$ has no existence for all $d_i \in I_o$

If $2d^2 = d_1^2 - d_2^2 \Rightarrow d_1^2 - 2d^2 = d_2^2 = 2d_2^2 - d_2^2$
$\Rightarrow$ WL of ($k = 2d^2$) = WL of ($k = d_2^2$) i.e. equality of WL for opposite kind of N-equation.
But as per 7.6.1, d & d_2 both being odd it is not possible. $2d^2 = d_1^2 - d_2^2$ form cannot exist.
$\Rightarrow$ w(x) for 2nd kind N-eq. having $k = 2I_o^2$ is always square-free function.
$\Rightarrow$ w(f(x)) is also square-free polynomials for $f(x) > 0$ with integer coefficients.

8.3 $2d_1^2 + 2d_2^2 = d_3^2 + d_4^2$ has no existence for all $d_i \in I_o$

Here, $2d_1^2 - d_3^2 = d_4^2 - 2d_2^2$
$\Rightarrow$ WL of ($k = d_3^2$) = WL of ($k = 2d_2^2$) i.e. equality of WL for opposite kind of N-equation.
d_1 & d_2 both being odd it is not possible.
$\Rightarrow$ w(x) for $k = 2I_o^2$ & k for 1st kind are non-intersecting.

8.4 $2v_1^2 + 2v_2^2 = d_1^2 + d_2^2$ & $2v^2 - 2d^2 = d_1^2 - d_2^2$ have no existence for all $v_i \in I_e$ & $d_i \in I_o$

Here also same logic is applicable.
$\Rightarrow$ w(x) for $k = 2I_e^2$ & for $k = I_o^2$ are always non-intersecting.
& w(x) for $k = 2I_e^2$ & for $k = 2I_o^2$ are always non-intersecting.

8.5 $E(2d_i^2 - d_j^2)$ has no existence where all $d_i, d_j \in I_o$ & > 1

Let us consider a relation $(d_1^2 - d_2^2)/(d_3^2 - d_4^2) = 2$ & squaring both sides we have,
$d_1^4 + d_2^4 - 2(d_1d_2)^2 = \{2(d_3^2 - d_4^2)\}^2 = v^2$ (say) $\Rightarrow d_1^2(2d_2^2 - d_1^2) = (d_2^2)^2 - v^2$
or, $(2d_2^2 - d_1^2) = \{(d_2^2)^2 - v^2\}/d_1^2$. As $d_2 \neq d_1$, RHS must be a square integer I_o^2 so that RHS can be written as $2I_o^2 - I_o^2 \Rightarrow d_1^2|\{(d_2^2)^2 - v^2\}$ and similarly, $d_2^2|\{(d_1^2)^2 - v^2\}$
$\Rightarrow (d_2^2)^2 - v^2 = \lambda d_1^2$ and $(d_1^2)^2 - v^2 = \lambda d_2^2 \Rightarrow$ on subtracting $\lambda = -(d_1^2 + d_2^2)$ which is absurd.
Hence, $2d_3^2 - d_1^2 = 2d_4^2 - d_2^2$ has no existence. But $d_1^2 - 2d_3^2 = d_2^2 - 2d_4^2$ exists. It indicates whenever there exists a relation like $(d_1^2 - d_2^2)/(d_3^2 - d_4^2) = 2$ it will produce equality of two wings under 2nd kind N-eq. only.
Or, it is simple logic from 8.2, that $2d_1^2 \neq d_2^2 - d_3^2$ & $2d_4^2 \neq d_5^2 - d_3^2 \Rightarrow 2d_1^2 - 2d_2^2 \neq d_2^2 - d_5^2$

In view of the above we can establish the following important corollaries.

Corollary 1: $d^2 = E(2x_i^2 - d_i^2)$, i = 1, 2, 3, ….. exists only for d = 1.
Corollary 2: In between any two $I_0(d_i^2, d_j^2)$ where $d_i, d_j > 1$, there cannot be any common element
Corollary 3: w(x) for a 1st kind N-eq. produces square integers either only for once or twice or not at all.
Corollary 4: w(2x) cannot produce any square integer.
e.g. for k = 1, $w(x) = 2x^2 - 1 \Rightarrow w(2x) = 8x^2 - 1$ is a square free function. In general, it is $8x^2 - d^2$
for $k = 3^2$, $w(x) = 2x^2 + 8x - 1 \Rightarrow w(2x) = 8x^2 + 16x - 1$ is a square free function
for $k = 19^2$, $w(x) = 2(x + p)^2 - 19^2$ where p = 13 so that at x = 1, $2(x + p)^2$ is just greater than 19^2.
x + p must be even. So replace x by 2x – 1 i.e. $2(2x + 12)^2 - 361$ i.e. $w(2x) = 8x^2 + 96x - 73$ is a square free function.
In first two examples w(x) is taken as per derivation under 1.3. but we can avoid derivation as per example 3 where attention is to be given regarding replacement of x by 2x or 2x – 1.
Corollary 5: $k = d^2 > 1$ where k doesn't belong to $\{d_i^2, d_j^2\}$ of $I_0(d_i^2, d_j^2)$ can be denoted by $I_0(d_i^2)$
$\Rightarrow$ all w(x) for such $k = d^2 \in I_0(d_i^2)$ produces square integer only for once simply by
$d^2 = (2.d^2 - d^2) = d^2(2.1^2 - 1)$ e.g. for $k = 3^2$, $w(x) = 2(3^2) - 3^2 = 2(x^2) - 3^2$.

So $x = 1$ i.e. leading set of $k = 3^2$ i.e. $27^2 + 36^2 = 45^2$.
For $2(x + p)^2 - d^2$ if $p = d - 1$, it will be the leading set.
$\Rightarrow$ for $k = 3^2$, $w(x + 1)$ is a square free function.
Corollary 6: if $w(x)$ is a square free quadratic function then it is obvious $w(g(x))$ where $g(x) > 0$ is a polynomial function with integer coefficients, will also be a square free polynomials of higher degree.
Corollary 7: $f(x, y) = \{2(2y - 1)\}^2 + (2x - 1)^2 \neq I_o^2$ as it fails to satisfy N-eq.

References:

1. Publication in IJSER – Aug-edition - 2013, vol-4, issue-8 (for content nos. 1, 2, 3, 5 with all sub-contents for ready reference)
2. Publication in IJSER – Aug-edition - 2013, vol-4, issue-8 (for content nos. 4, 6 with little addition of examples & corollaries 6.9, 6.10, 6.11)
3. Publication in IJSER – Sept-edition - 2015, vol-6, issue-9 (7.1 to 7.5 for ready reference)
4. New contribution for 7.6.1, 7.6.2, 7.6.3, 8.1, 8.2, 8.3, 8.4, 8.5

Conclusion: With further development of Wings-theory, I must take the privilege to mention the exceptional sequence of square integers produced with respect to my earlier theorem no. 10.5 published in 'IJSER' of vol-7, issue-8, Oct-edition, 2016. According to this theorem if $N = x(x + 1)$ then $N + u^2 \neq I^2$ where u is any factor of x or $(x + 1)$ including unity. $\Rightarrow N/u + u \neq I^2$ when $u \in I^2$ But if $u = x + 1$ then by virtue of abc-equation (Ref. 7.1 of this article) following sequence of exceptional cases to produce square integers is found to be quite justified.
Say, $N = (5^2 - 1).5^2$ as per 7.1, $\Rightarrow (5^2 - 1).5^2 + (5^2)^2 \neq I^2$ i.e. $(5^2 - 1) + 5^2 \neq I^2$ as per 10.5 of earlier publication. But $(5^2 - 1) + 5^2$ i.e. $2.5^2 - 1 = 7^2$. Similarly, for $N = (169^2 - 1).169^2$, $N + (169^2)^2$ i.e. $(169^2 - 1) + 169^2$ i.e. $2.169^2 - 1 \neq I^2$. But it is equal to 239^2. Similarly for the next sequence where $N = (195025^2 - 1).195025^2$, $N + 195025^2$ i.e. $2.195025^2 - 1 \neq I^2$ but it is equal to 275807^2 and so on. $\Rightarrow$ if $N = (2y + 1)(2y + 2)$ it never fall under that exceptional sequence as smaller part $(2y + 1)$ is odd.

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